2. The geometry which is involved is of a novel but simple character. It is entirely two-dimensional so long as patterns in two dimensions are in view. Two figures are looked upon as identical only when they can be made to coincide by movements in the plane. If two figures be such that one is the image of the other, but not identical with it, and one be denoted by a, the other is conveniently denoted by Ia, and, when necessary, the position of the line which determines the position of the image will be stated.

If a point α be joined to a fixed point O in the plane and the line αO be produced an equal distance to α'; α, α', are the point images of one another (or are in point symmetry) with respect to the point O.

Any locus of points α has a point image with respect to O which is the locus of the corresponding points α'.

The point image of a figure with respect to any point in its plane is an identical figure rotated through two right angles about an axis perpendicular to the plane.

The figure and the point image are conveniently denoted by

$$a, \; Pa,$$

the point O being specified when it is necessary to fix the relative positions of the two figures.

Generalities concerning Repeats.

3. It is convenient to bring together some well known facts, along with others which may not be well known or may be new.

The only regular polygons which are repeats are the equilateral triangle, the square, and the regular hexagon.

These are important bases for use in the method of construction which will be presented.

Every parallelogram is a repeat.

Since any parallelogram can be divided into two identical triangles by drawing either diagonal, it follows that

Every triangle is a repeat.

4. *Every Quadrilateral is a Repeat.*

This valuable result appears hitherto to have escaped the notice of geometricians.† It is fundamental in the subject, and is obvious to any geometrician almost as soon as it is stated. It follows readily from the fact that the sum of the interior angles is equal to four right angles.

Every hexagon in which each pair of opposite sides involves lines which

Cambridge University Press. It is there dealt with as affording a mathematical recreation, and is not seriously considered from a scientific point of view.

† The result for a *convex* quadrilateral is given as Example 14, p. 86, of Sommerville's 'Non-Euclidean Geometry,' 1914.

are equal and parallel is a repeat. This theorem is obvious directly a few hexagons are assembled. In this case the sum of three alternate interior angles is equal to four right angles.

Observe also that if we join two opposite angular points of such a hexagon, we divide it into two identical quadrilaterals, and that any quadrilateral is obtainable in this way. Since the hexagon is a repeat, its bisection into identical quadrilaterals establishes the quadrilateral as a repeat. Conversely, if we assume the repeat property of the quadrilateral, that of the hexagon is established.

5. *The Principle of Dissection.*

We have found that the parallelogram and the special hexagon above defined give each rise to a new repeat by bisection into identical figures. We have here a general principle. Whenever it is possible to dissect a repeat into a number of identical figures, the figure so derived is a repeat.

6. *The Principle of Composition.*

When we assemble a given repeat, we can in general draw a boundary enclosing two or more repeats, such that the figure enclosed by the boundary is a repeat. Thus, from the quadrilateral repeat we derive the special hexagon repeat. Many examples of composed repeats present themselves later. The principle is the converse of that of "dissection."

7. *Nature of an Assemblage of Repeats.*

When a repeat is assembled, it *may* always appear in the same orientation. Such is the case with the square, the regular hexagon, and the special hexagon connected with the quadrilaterals. Such an assemblage may be said to exhibit one aspect of repeat.

On another hand, if we assemble the equilateral triangle, the general quadrilateral and an infinite number of other repeats, we find that the assemblage exhibits two aspects of the repeat. Other assemblages, according to the nature of the repeat, involve three, four, or six aspects.

This is not the whole story, because some repeats can be assembled in more than one way, and these ways may exhibit different numbers of aspects. As an example, it will be shown that one category of repeats is such that every member of it may be assembled in three different ways, exhibiting one, two, and four aspects respectively. The nature of the assemblage in regard to aspect is an important feature of the classification. At the present stage it need only be remarked that the equilateral triangle is the only one of the three repeating regular polygons that *cannot* have fewer than two aspects in assemblage. In general, a repeat derived by "dissection" has more aspects when assembled than the repeat dissected; and, conversely, the repeat derived by "composition" has fewer aspects than the repeat from which it is derived.

8. *The Principle of Absorption.*

To explain, we make an assemblage of equilateral triangles, a tessellation, and number the triangles, which have the same aspect, 1, and the remaining triangles are numbered 2. Then we see that every triangle numbered 1 or 2 is adjacent to three triangles numbered 2 or 1 respectively. Dissect each of the triangles numbered 2 by lines drawn from the centre to the vertices. Add to the triangles nnmbered 1 the adjacent thirds of each of the adjacent triangles; thus forming an assemblage of regular hexagons at the expense of the triangles numbered 2, which have all been absorbed by the triangles numbered 1.

This is the simplest illustration of the principle. It may be at once generalised for the equilateral triangle by taking any trisection at pleasure into three identical figures. The square, the regular hexagon, and many other simple forms may be treated in a similar manner.

9. *The Principle of Line Transformation.*

It is convenient to regard the equilateral triangle as the simplest repeat. The simplest way (not the only one) of assembling it is to cause each side to touch, throughout its entire length, the side of another triangle. The method of transformation, as applied to such an assemblage, consists in transforming each triangle to some other shape, the same in each case, so that the flat space is still filled up without leaving interstices. When this is carried out the triangle, as transformed in shape, is established as a repeat.

To effect such a transformation we divide the triangle into three compartments by straight lines drawn from the geometrical centre to the vertices, and suppose the three identical compartments to be numbered 1, 2, 3, such numbers being in ascending order of magnitude when read counter-clockwise about the centre. It is convenient to regard these numbers as symbolising colours.

It is not quite a trivial remark that these triangles may be assembled so that the colours 1, 2, 3 are adjacent to the colours 1, 2, 3 respectively. The fact is established by the diagram, fig. I, which may be extended indefinitely.

If the colours 1, 2, 3 be the same we have the ordinary tessellation composed of identical equilateral triangles; we have nothing new. But if the colours be not the same we have an assemblage subject to a certain *contact system* which may be briefly denoted by

1 to 1, 2 to 2, 3 to 3, or 11, 22, 33.

We have a *new kind of repeat.*

10. Moreover the contact system may be varied. In the present instance there is only one other essential variation, viz., the contact system

1 to 1, 2 to 3, or 11, 23.

This is established by the diagram, fig. II, which can be extended indefinitely. We have again a new repeat because the assemblage is subject to another system of contact. The diagrams show that the new repeats exhibit two and six aspects respectively.

There may be identities between the colours which must be subjected to examination, but *always* the contacts are of two, and only two, natures:—

(i) The adjacent compartments are similarly coloured.
(ii) „ „ „ differently „

11. We next seek to distinguish the three compartments of the triangle, not by different colours, but by altering the shapes of the compartments differently. The object is to abolish the colours by transforming the colours into shapes.

The First Nature of Contact.

The adjacent compartments are similarly coloured.

In the left-hand diagram of fig. III, the common boundary of the compartments coloured 1 is a straight line; in the right-hand diagram the colour 1 has disappeared and the adjacent compartments have a common zig-zag boundary which is of such a shape that the two triangles have been transformed so that they remain of the same shape.

We have thus a repeat of a new shape and involving two colours. The assemblage is subject to the contact system

22, 33.

The enquiry is now into the law which controls the shape which replaces the colour.

If AB be the initial boundary we see that the new boundary line which connects the points AB must not be changed in orientation by a rotation through two right angles about a perpendicular, through the mid-point of AB, to the plane of the paper. In other words the point image of the new boundary with respect to the mid-point of AB must be identical with that boundary both in shape and orientation.

If we desired to treat each of the three colours in the same way, we could restrict the new boundary so as to lie entirely within the rhombus originally formed by the two compartments, coloured 1, when in contact. But this is not always to be desired, and other bases than the equilateral triangle come into view, and other conditions. The principles which will now be set forth

are applicable to all bases and to various conditions, so that it is not convenient to restrict in any way the delineations of boundaries which explain and exemplify the principles, but rather to leave them to be subsequently modified so as to conform to requirements which may present themselves.

Suppose that the dotted line AB denotes the original boundary of the compartment coloured 1 and that the base piece extends below AB, the shape of the piece and of the compartment not being at present before us.

Change the boundary to the zig-zag A*b*O*a*B where O is the mid-point of AB. If A*b*O*a*B be its own point image with regard to O it follows that the part B*a*O above AB is the point image of the part A*b*O below AB, fig. IV.

In general we can draw any system of closed curves above AB and the corresponding closed curves, point images of the former, below AB and the required transformation will be effected.

The closed curves above AB determine additions to the base piece while those below AB determine corresponding pieces which must be cut away from the base. Examples are given in fig. XV.

Shapes thus defined fit together and are to be employed in substitution of the colour. Each colour is treated on the same principle but care must be taken that the several transformations are compatible with one another. More upon the subject of compatibility is postponed to a later stage in the paper.

12. *Symmetry of Shape.*

Having a symmetrical calculus in view we must distinguish between the transformation patterns which are symmetrical about the line which bisects the line AB at right angles and those which do not possess this symmetry.

The symmetrical shape will be denoted by

$$S_1$$

the unit suffix denoting that the first nature of contact (that between identical colours) is in view.

Different symmetrical shapes are denoted by

$$S_1, \quad S_1', \quad S_1'', \ldots .$$

The unsymmetrical shapes are denoted by

$$U_1, \quad U_1', \quad U_1'', \ldots .$$

It must be noticed that the symmetrical shape has the same shape as its image with respect to a line perpendicular to AB in the plane of the paper so that

$$S_1 \equiv IS_1$$

but

$$U_1 \not\equiv IU_1.$$

The symbol I is not to be regarded as an operator. The notation merely means that the patterns U_1, IU_1 are related to one another in a certain manner.

Briefly to resume, we may say that, denoting the base figure by a shaded area, we may take any system of shaded areas exterior to and above AB (additions to the base) and a corresponding system of unshaded areas within the area of the base figure (gaps in the base piece), the boundaries of the shaded and unshaded areas being point images of one another with respect to the mid-point of AB.

On the suggestion of Mr. G. T. Bennett we call repeats which have gaps in them "stencil" repeats, and those which involve several separate pieces, relatively fixed in position, "archipelago" repeats. Both of these come into view in the transformation due to the first nature of contact.

13. *The Second Nature of Contact.*

Here a colour 2 is invariably adjacent to a colour 3. We require a transformation such that it is mechanically impossible for the compartments so coloured to be in juxtaposition in any other manner.

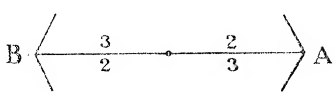
It is clear that if we take, in respect of the compartment coloured 2, any system of closed curves above AB and any system below AB within the base as a definition of its transformation we at the same time completely determine the transformation pattern that must be associated with the colour 3.

We always view the transformation pattern from the inside of the base looking outwards, so that the closed curves which define the shape associated with the colour 3 are derived from those which define the shape associated with the colour 2, by taking the point image of the latter with regard to the mid-point of AB, fig. V.

Suppose then that we are given, by shaded and unshaded areas, the transformation pattern of the colour 2, we take the point images of the contours of the added and subtracted areas with respect to the mid-point of AB, shading those areas which then become exterior to the base, and leaving unshaded those which become interior.

We thus arrive at the transformation pattern for the colour 3. Examples will be seen in fig. V.

The two transformation patterns are connected in the same way as the exterior and interior patterns arising from the first nature of contact. The first nature of contact can, in fact, be exhibited as two contacts of the second nature, one on each of two halves of the original boundary.



The only condition of transformation is that the two transformation patterns, determined as above, must be *different*. In the contrary case they would belong to the first nature of contact.

Put in another way, we have merely to take any convenient set of closed curves, not belonging to the first nature of contact, to replace the colour 2; and then the point image process produces the closed curves which are to be taken to replace the colour 3. Examples are shown in fig. XV.

14. For the purpose of the symmetrical calculus these patterns are separable into three classes:—

(i) Those which are symmetrical about a straight line at right angles to AB and passing through its mid point.

Such will be denoted by

$$S_2, \quad S_2', \quad S_2'', \ldots .$$

(ii) Those which do not belong to the class (i), and, further, are not symmetrical about AB. Such will be denoted by

$$U_2, \quad U_2', \quad U_2'', \ldots .$$

(iii) Those which do not belong to the class (i) but are symmetrical about AB. Such will be denoted by

$$V_2, \quad V_2', \quad V_2'', \ldots .$$

the suffix 2 always denoting that the second kind of contact is in view.

Examples of the three classes are given in fig. XV.

16. The symbol I being as defined above

$$IS_2 \equiv S_2, \quad IU_2 \not\equiv U_2, \quad IV_2 \not\equiv V_2.$$

For a 2 to 3 contact, if S_2, U_2, V_2, be patterns associated with the colour 1, PS_2, PU_2, PV_2, denote the patterns to be associated with the colour 2.

In the shapes, symbolised by V_2,

$$PV_2 \equiv IV_2.$$

15. *The Construction of Symmetrical Repeats.*

A repeat may be symmetrical—

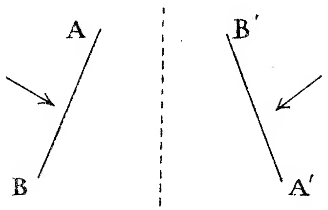
(i) About an axis in the plane of the paper.

(ii) By being unchanged by certain rotations about an axis perpendicular to the plane of the paper.

In the case of (i) there are definite principles which it is useful to recognise at once.

The axis of symmetry will usually be either a straight line passing through an angular point of the base polygon or a straight line which is perpendicular to a side of the base polygon, at the mid-point of the side.

Suppose that AB, A′B′ are two equal sides of the base polygon, and symmetrically situated with regard to the axis $\alpha\beta$.



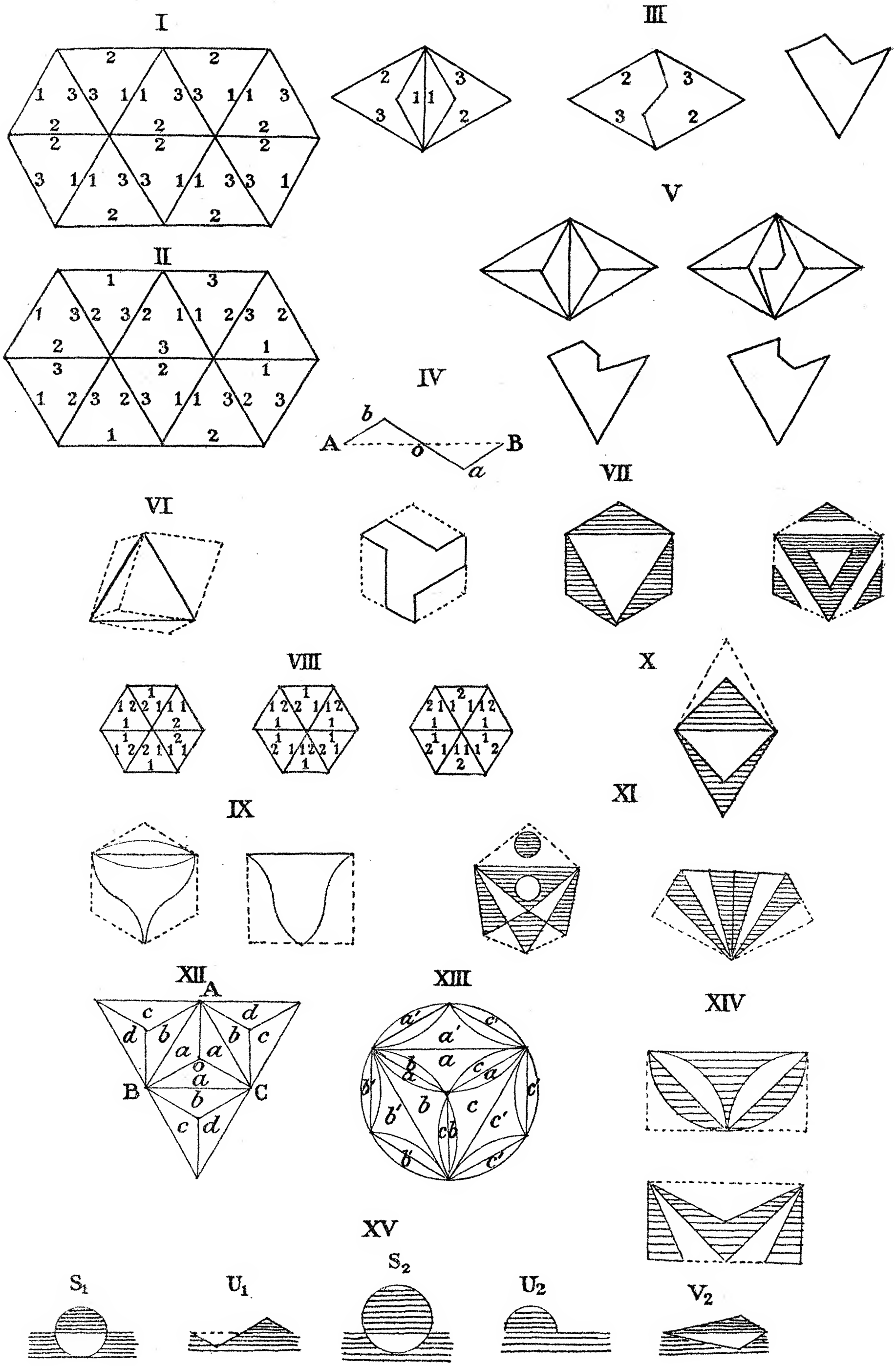
The boundary patterns are viewed in the direction indicated by the arrows. We obtain symmetry by taking

for AB	S_1,	for A′B′	S_1,
,,	U_1,	,,	IU_1,
,,	S_2,	,,	S_2,
,,	U_2,	,,	IU_2,
,,	V_2,	,,	$PV_2 \equiv IV_2$.

In the case of (ii) there is no difficulty in obtaining symmetry. The matter will be dealt with when the various bases are brought into view.

16. We are now in a position to resume the discussion of the repeats which may be derived from an equilateral triangular base. It is, as above remarked, essential to secure compatibility between the transformation patterns. If we take any point O in the area of the triangle (fig. VI) and join it to the vertices, we divide it into three parts which may be associated with the corresponding sides, BC, CA, AB. We may take the triangles OBC, OCA, OAB for the coloured compartments. The point image of O with respect to the mid-points of the three sides being the points A′, B′, C′, we may take the parallelograms OA′, OB′ OC′ for the areas within which the transformation patterns are to be restricted, where the contact system only involves contacts of the first nature. (Readers will supply letters to fig. VI.)

To obtain a generalisation, observe that we may proceed in a similar manner in regard to *any* division of the triangle into three parts. In particular, we may associate the parallelograms OA′, OB′, OC′, above, in any manner with the sides, BC, CA, AB. Since the triangular base has symmetry of the third order, the number of different shapes thus obtained from the same division of the triangle and the same transformations will be two, viz., the number of permutations of three different things arranged in circular order. If the base is without symmetry, there will be the full number, six, of different repeats derived from the same division of the triangle and the same transformations.


I
II
III
IV
A
B
a
b
o
V
VI
VII
VIII
IX
X
XI
XII
A
B
C
XIII
XIV
XV
S1
U1
S2
U2
V2

17. *The First System of Contact.*

There are three cases to consider corresponding to the colouring of the compartments in the ways

(*a*) 1, 1, 1, (*b*) 1, 1, 2, (*c*) 1, 2, 3.

In order to exhibit some of the results, we may take the simplest trisection of the triangle to guide the transformations. For (*a*) three examples are given in fig. VII. They are examples of "block," "stencil," and "archipelago" repeats respectively.

18. The stencil example is of particular interest because it can be used as an actual stencil to produce the equilateral triangle repeat which is the base of the system. This observation may appear at first sight to be trivial but in reality supplies the first glimpse of a principle of great importance. It is in general possible to draw a contour surrounding any given repeat in such wise that the area inside the contour, which is not part of the original repeat, itself is a repeat. In the present instance the contour referred to is a regular hexagon and the illustration by its shaded and unshaded portions exhibits two repeats.

The duality arises in the following way. If in a rhombus, to which the pattern which replaces the colour is restricted, we draw the system of closed curves according to rule, it is clear that those areas of the rhombus which have not been enclosed possess the same property, viz., the boundaries of those areas which are inside the boundary of the base are the point images of those which are outside that boundary. It follows that the boundary patterns which replace the colours always present themselves, *quâ* the rhombus, in pairs. Moreover, for *any* division of the triangle base into parts and for *any* association of these parts with the sides of the triangle, the transformation patterns invariably present themselves in pairs.

It will be observed also that the external boundary, in the present instances a regular hexagon, also by itself determines a repeat so that in reality each figure exhibits three repeats. In connection with this circumstance there is a theory of some elegance which will be dealt with later in the paper. It is involved in one of the fundamental theorems of the geometry of repeats.

The repeats of this category, viz., 1, 1, 1, can be assembled in only one manner and two aspects are always exhibited. The category embraces an infinite nnmber of repeats.

19. For *b* (1, 1, 2) there are three ways of assembling exhibiting 6, 2, 6 aspects of the repeat respectively, fig. VIII.

In the transformation of the colours 1, 2 we select any two of the patterns symbolised by S_1, U_1 and we may take for the point O which determines the areas within which the patterns may be drawn, any point in the area of the

triangle. An example of repeats of this category is shown in fig. X the external boundary which determines a second repeat being shown dotted.

20. For c (1, 2, 3) we select three different patterns and by using both U_1 and IU_1 it is easy to secure symmetry. Examples are given in fig. IX.

21. *The Second System of Contact*, 1 to 1, 2 to 3 or 11, 23.

There is only one way of constructing a symmetrical repeat, viz., by taking S_1 for 1, V_2 for 2, and PV_2 for 3.

Take the point O upon a perpendicular and join it by straight lines to the other two vertices. The perpendicular being the axis of symmetry the compartments coloured 2, 3 are on either side of it and it bisects the compartment coloured 1. Construct the images of the compartments coloured 2, 3 in the corresponding sides and the point image (in the present instance this is the same as the image) of that coloured 1, thus forming an exterior polygon, viz., a hexagon. It is clear that if we construct any V_2 pattern within the prescribed area the portion of that area which does not belong to the pattern will also be a V_2 pattern. Hence the exterior polygon may be employed to form a second repeat by the process of cutting away from it any repeating pattern which has been constructed according to rule. In particular we may cut away the triangular base.

We may also vary the trisection of the triangle in the manner described in Nos. 5, 22, and proceed to an external contour by taking two images and one point image as above. Examples are shown in figs. X, XI.

22. *Concerning the Principles of Dissection and Composition.*

There is a point that is made conveniently at this stage of the research. The equilateral triangle yields an infinite number of repeats by trisection into three identical parts, the lines of section being straight or curved and drawn through the geometrical centre.

Conversely any one of these repeats may be assembled so as to form other repeats. Now the three identical pieces into which the triangle has been dissected may be put together so as to form a repeat in *more than one way.* Thus looking at the diagram, fig. XII, the repeat marked a may be assembled as to form the original equilateral triangle. If we take the point images of the repeats thus placed with regard to the mid-points of the adjacent sides, we obtain the stencil repeat constituted by the areas marked b. Further, if we take the point images of the areas AoA, BoC, CoA, with regard to the mid-points of CA, AB, BC, respectively, we arrive at the archipelago repeat constituted by the areas marked c; and if with regard to the mid-points of BC, CA, AB, respectively, the archipelago repeat constituted by the areas marked d. We can, therefore, in every case assemble the repeat in four different ways so as to form another repeat. The proof of the above follows at

once from the rules of transformation that have been given. This point will appear again at a later stage.

23. *The Classification of Repeats.*

The foregoing pages suggest a method of classifying repeats depending upon:

(1) the simple geometrical form taken as base;

(2) the contact system which guides the transformation from colour to shape;

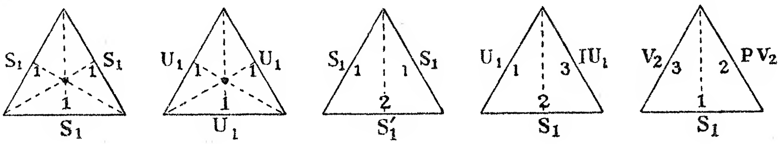
(3) the nature of the colouring of the compartments of the base;

(4) In the case of symmetrical repeats the transformation is indicated by the symbols S_1, U_1, S_2, U_2, V_2.

We will denote the equilateral triangle base by the letters T. E. and the contact systems 11, 22, 33; 11, 23 by (i), (ii), respectively, so that the classification is

T.E. (i) 1, 1, 1; 1, 1, 2; 1, 2, 3.
T.E. (ii) 1, 2, 3.

For symmetrical forms we have the five categories:—



24. *The Theory of Complementary Repeats.*

It is evident from the preceding discussion that every repeat is associated with an infinite number of complementary repeats which are determined by the division of the base into three (n) portions. One or both (some or all) of two ($n-1$) of the portions may have no area and therefore be non-existent.

On the other hand the dissection of the triangle (base polygon) may be into any number of distinct areas, and these may be grouped into three (n) lots, constituting the three (n) portions; moreover, any number up to two ($n-1$) of these lots may be non-existent.

To illustrate this point consider the circular stencil repeat derived from the equilateral triangle, fig. XIII. The division of the triangle is into nine areas grouped into three lots marked a, b, c, respectively. The point images of these with respect to the corrresponding sides are marked a', b', c', respectively. In regard to the upper side the areas marked a and a' are available for the transformation pattern of type S_1, and similarly for the other sides the areas marked b, b'; c, c'.

The pattern of the repeat is given by the *small* areas a', b', c', and the large areas a, b, c.

The exterior contour which determines the complementary repeat is composed of the boundary lines of the small areas a', b', c', and the *curved* boundaries of the larger areas a', b', c'. The contour consequently consists of seven distinct closed boundaries of which six enclose the small areas a', b', c'.

The complementary repeat is constituted by the collection of areas enclosed by the seven contours which do not belong to the first repeat. There are no such areas enclosed by the six small contours, so that *effectively* the exterior contour is determined by the curved boundary of the large areas a', b', c', and we are led to the complementary repeat constituted by the *large* areas a', b', c' and the *small* areas a, b, c.

It may be noted also that the third repeat (see fig. XIII) consists of the whole of the areas marked with the letters a, b, c, a', b', c', and is identical with the circular repeat that is derivable from a regular hexagonal base. This is a stencil form and is the nearest approach to a circular block repeat that it is possible to construct.

The subject of complementary repeats in general is considered again in Part II.

25. *The Isosceles Right-Angled Triangle Base T. R.*

We can divide the triangle into three parts in any manner, but for present purposes it is convenient to effect the division by joining the vertices to some point O upon the perpendicular drawn from the right angle on to the base. For most purposes it suffices to take the point O at the centre of gravity of the triangle. We number the compartments 1, 2, 3 in counter-clock-wise order and first consider the contact system

11, 22, 33.

Assemblage can be made in one way; but when the colours 2, 3 are identical a second way of assembling is found which exhibits four aspects instead of two.

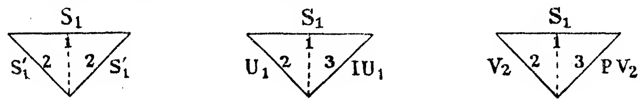
With the contact system

11, 23,

there is only one way of assembling and this exhibits four aspects. The classification is

T.R. (i) 1, 2, 2; 1, 2, 3.
T.R. (ii) 1, 2, 3.

There are three symmetrical repeats :—



Some examples are given in fig. XIV.

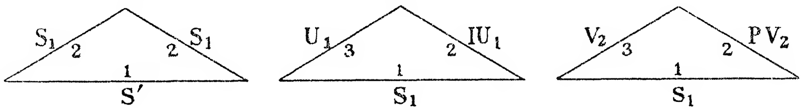
The point O which determines the dissection of the base is conveniently taken either at the middle point of the hypothenuse or at the centre of gravity. The exterior boundary which determines a second repeat is shown in dotted lines.

26. The isosceles triangle with vertical angle 120°, T. $\frac{2}{3}\pi$ can be assembled either on the contact system 11, 22, 33, or on 11, 23.

The classification is

T. $\frac{2}{3}\pi$. (i) 1, 2, 2; 1, 2, 3.
T. $\frac{2}{3}\pi$. (ii) 1, 2, 3.

There are three symmetrical repeats:—



The point O may be suitably placed either at the vertex or at the mid point of the long side.

Correlation between Arrays in a Table of Correlations.

By C. Spearman.

(Communicated by Prof. L. N. G. Filon, F.R.S. Received December 28, 1921.)

Two recent papers in these Proceedings have dealt with certain problems of probability which are of very great importance for psychology in particular, but in themselves are quite general.* As both papers made reference to some of my own, the following considerations may not only be a further positive contribution to the topic, but incidentally serve to clear up some misunderstandings.

Suppose that the product moment coefficients of correlation have been determined between any set of variables $a, b, c, d, \ldots z$, and have been set forth in a square Table thus:—

	a	b	c	.	.	z
a		r_{ab}	r_{ac}	.	.	r_{az}
b	r_{ab}		r_{bc}	.	.	r_{bz}
c	r_{ac}	r_{bc}		.	.	r_{cz}
.	.	.	.	.	.	.
.	.	.	.	.	.	.
z	r_{az}	r_{bz}	r_{cz}	.	.	

* Maxwell Garnett, 1919; G. Thomson, 1919.